Section Handout 1

Problem One: Finding Flaws in Proofs

The following proofs all contain errors that allow them to prove results that are patently false. For each proof, identify at least one flaw in the proof and explain what the problem is, then give a counterexample that demonstrates why the error occurs. In each case, **make sure you understand what logical error is being made**. The mistakes made here are extremely common.

Theorem: If n is even, then n^2 is odd.
Proof: By contradiction; assume that n is odd but that n^2 is even. Since n is odd, $n = 2k + 1$ for some integer k. Thus $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. This contradicts our earlier claim that n^2 is even. We have reached a contradiction, so our initial assumption was wrong. Thus if n is even, n^2 is odd. ■

Theorem: For all sets A and B, $A \cup B = A$.
Proof: By contradiction; assume that for all sets A and B, $A \cup B \neq A$. So consider $A = \emptyset$ and $B = \emptyset$; then $A \cup B = A$. This contradicts our earlier claim that $A \cup B \neq A$ for all A and B. We have reached a contradiction, so our initial assumption was wrong. Thus for any sets A and B, $A \cup B = A$. ■

Theorem: If $C \subseteq A \cup B$, then $C \subseteq A$.
Proof: By contrapositive. We prove that if C is not a subset of $A \cup B$, then it is not a subset of A. Since C is not a subset of $A \cup B$, there is some $x \in C$ such that $x \notin A \cup B$. Since $x \notin A \cup B$, $x \notin A$ and $x \notin B$. Thus $x \in C$ but $x \notin A$, and so C is not a subset of A. ■

Problem Two: Properties of Sets

Below are three claims, some of which are always true, some of which are always false, and some of which are sometimes true and sometimes false, depending on what sets are chosen. For each statement, if it is always true, prove it. If it is always false, prove it. If it is sometimes true and sometimes false, provide an example for which it is true and an example for which it is false.

To prove that two sets are equal, remember that you need to show that any element of the first set must also be an element of the second set and vice versa. Recall that this is equivalent to showing that the two sets are each subsets of one another. Also, it is **not** sufficient to use Venn diagrams or any other informal reasoning here. You need to formally prove each result.

i. $A \subsetneq B$ and $A \subsetneq C$, then $A \subsetneq B \cap C$.

ii. If $A \subsetneq \emptyset$, then $137 \in A$.

iii. If $x \in A$, then $x \in \wp(A)$

Problem Three: Quadratic Equations

A *quadratic equation* is an equation of the form $ax^2 + bx + c = 0$. A *root* of the equation is a real number x satisfying the equation.

Recall from lecture that a rational number is one that can be written as p/q for integers p and q where $q \neq 0$ and p and q have no common divisor other than 1.

i. Prove that mn is odd iff m is odd and n is odd.

ii. Prove, by contradiction, that $x^2 + 3x + 1$ has no rational roots. Be sure to explicitly state what assumption you are attempting to contradict. As a hint, if the rational solution is p/q, consider what happens if both p and q are odd and what happens if exactly one of p and q is odd. (Why can't both p and q be even?)